

ALGEBRAIC PROPERTIES OF \mathfrak{G} -GROUPOIDS

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Abstract. The main purpose of this paper is to give a new definition for the notion of group-groupoid. Also, several basic properties of group-groupoids are established.¹

1 Introduction

There are basically two ways of approaching groupoids. The first one is the category theoretical approach. The second one is algebraically considering them as a particular generalization of the structure of group. Groupoids are like groups, but with partial multiplication; i.e., only specified pairs of elements can be multiplied and inverses with respect to the multiplication exist for each element.

The groupoid was introduced by H. Brandt [Math. Ann., **96**(1926), 360-366] and it is developed by P. J. Higgins in [6].

The notion of group-groupoid was defined by R. Brown and Spencer in the paper [4]. A group-groupoid is viewed as a groupoid object in the category of groups ([1]).

The groupoids, group-groupoids and their generalizations (topological groupoids and topological group-groupoids, Lie groupoids and Lie group-groupoids etc.) are mathematical structures that have proved to be useful in many areas of science (see for instance [2, 16, 12, 7, 5, 8, 13]).

The paper is organized as follows. In Section 2 we present some basic facts about groupoids. In Section 3 we present the notion of group-groupoid as given in [3] and [4]. We prove a main theorem for characterization the group groupoids. This is used for give a new definition for the concept of group-groupoid. Applying these facts we establish some important properties in the category of group-groupoids.

2 Preliminaries about groupoids

We recall the minimal necessary backgrounds on groupoids ([12, 16]).

A way to think of a groupoid is to say that a *groupoid* is a small category in which every morphism is an isomorphism ([6, 1]). Thus, a groupoid consists of:

- two sets G and G_0 called the *set of arrows (or elements)* and a *set of objects* (or the *base*) respectively, together with maps $\alpha, \beta : G \rightarrow G_0$, $\varepsilon : G_0 \rightarrow G$ such that $\alpha \circ \varepsilon = \beta \circ \varepsilon = Id_{G_0}$;
- if $x, y \in G$ and $\beta(x) = \alpha(y)$, then a *product* xy exists such that $\alpha(xy) = \alpha(x)$ and $\beta(xy) = \beta(y)$, and this product is associative;
- $\varepsilon(u) \in G$ (denoted with 1_u) for $u \in G_0$, act as identities, and
- each $x \in G$ has an inverse $x^{-1} \in G$ with $\alpha(x^{-1}) = \beta(x)$, $\beta(x^{-1}) = \alpha(x)$, $xx^{-1} = 1_{\alpha(x)}$ and $x^{-1}x = 1_{\beta(x)}$.

In the following definition we describe the groupoid as algebraic structure.

¹AMS classification: 20L13, 20L99.

Key words and phrases: groupoid, group-groupoid.

Definition 2.1. ([12]) A *groupoid* G over G_0 is a pair (G, G_0) of sets endowed with two surjective maps $\alpha, \beta : G \rightarrow G_0$ (*source* and *target*), a partially *multiplication* $m : G_{(2)} := \{(x, y) \in G \times G \mid \beta(x) = \alpha(y)\} \rightarrow G$, $(x, y) \mapsto m(x, y) := x \cdot y$, ($G_{(2)}$ is the *set of composable pairs*), an injective map $\varepsilon : G_0 \rightarrow G$ (*inclusion map*) and a map $i : G \rightarrow G$, $x \mapsto i(x) := x^{-1}$ (*inversion*), which verify the following conditions:

- (G1) (*associativity*): if $(x, y) \in G_{(2)}$ and $(y, z) \in G_{(2)}$, then so $(x \cdot y, z) \in G_{(2)}$ and $(x, y \cdot z) \in G_{(2)}$, and the relation, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ is satisfied;
- (G2) (*units*): for each $x \in G$ it follows $(\varepsilon(\alpha(x)), x) \in G_{(2)}$, $(x, \varepsilon(\beta(x))) \in G_{(2)}$ and $\varepsilon(\alpha(x)) \cdot x = x = x \cdot \varepsilon(\beta(x))$;
- (G3) (*inverses*): for each $x \in G$ it follows $(x^{-1}, x) \in G_{(2)}$, $(x, x^{-1}) \in G_{(2)}$ and $x^{-1} \cdot x = \varepsilon(\beta(x))$, $x \cdot x^{-1} = \varepsilon(\alpha(x))$. \square

We write sometimes xy for $m(x, y)$, if $(x, y) \in G_{(2)}$. Whenever we write a product in a given groupoid, we are assuming that it is defined.

The element $\varepsilon(\alpha(x))$ (resp., $\varepsilon(\beta(x))$) is called the *left* (resp., *right unit*) of x ; $\varepsilon(G_0)$ is called the *unit set*; x^{-1} is called the *inverse* of x .

For a groupoid we use the notation $(G, \alpha, \beta, m, \varepsilon, i, G_0)$ or (G, G_0) ; $\alpha, \beta, m, \varepsilon, i$ are called the *structure functions* of G . For each $u \in G_0$, the set $\alpha^{-1}(u)$ (resp., $\beta^{-1}(u)$) is called α -*fibre* (resp., β -*fibre*) of G at u . The map $(\alpha, \beta) : G \rightarrow G_0 \times G_0$ defined by $(\alpha, \beta)(x) := (\alpha(x), \beta(x))$, $(\forall) x \in G$ is called the *anchor map* of G . A groupoid is *transitive*, if its anchor map is surjective.

For any $u \in G_0$, the set $G(u) := \alpha^{-1}(u) \cap \beta^{-1}(u)$ is a group under the restriction of the multiplication, called the *isotropy group at u* of the groupoid (G, G_0) .

If $(G, \alpha, \beta, m, \varepsilon, i, G_0)$ is a groupoid such that $G_0 \subseteq G$ and $\varepsilon : G_0 \rightarrow G$ is the inclusion, then we say that $(G, \alpha, \beta, m, i, G_0)$ is a G_0 -*groupoid* or a *Brandt groupoid*.

In the following proposition we summarize some basic properties of groupoids obtained directly from definitions.

Proposition 2.1. ([11]) *If $(G, \alpha, \beta, m, \varepsilon, i, G_0)$ is a groupoid, then:*

- (i) $\alpha(xy) = \alpha(x)$ and $\beta(xy) = \beta(y)$ for any $(x, y) \in G_{(2)}$;
- (ii) $\alpha(x^{-1}) = \beta(x)$ and $\beta(x^{-1}) = \alpha(x)$, $(\forall) x \in G$;
- (iii) $\alpha(\varepsilon(u)) = u$, $\beta(\varepsilon(u)) = u$, $\varepsilon(u) \cdot \varepsilon(u) = \varepsilon(u)$, $(\varepsilon(u))^{-1} = \varepsilon(u)$, $(\forall) u \in G_0$;
- (iv) if $(x, y) \in G_{(2)}$, then $(y^{-1}, x^{-1}) \in G_{(2)}$ and $(x \cdot y)^{-1} = y^{-1} \cdot x^{-1}$;
- (v) $\varphi : G(\alpha(x)) \rightarrow G(\beta(x))$, $\varphi(z) := x^{-1}zx$ is an isomorphism of groups.
- (vi) if (G, G_0) is transitive, then all isotropy groups are isomorphic. \square

Applying Proposition 2.1, it is easily to prove the following proposition.

Proposition 2.2. ([11]) *The structure functions of a groupoid $(G, \alpha, \beta, m, \varepsilon, i, G_0)$ verifies the following relations:*

$$\alpha \circ i = \beta, \quad \beta \circ i = \alpha, \quad i \circ \varepsilon = \varepsilon \quad \alpha \circ \varepsilon = \beta \circ \varepsilon = Id_{G_0} \quad \text{and} \quad i \circ i = Id_G. \quad \square$$

A subgroupoid of the groupoid (G, G_0) is a subcategory (H, H_0) of (G, G_0) such that (H, H_0) is itself a groupoid. A morphism between two groupoids is essentially a functor. These concepts are described in the following definition.

Definition 2.2. ([9]) Let $(G, \alpha, \beta, m, \varepsilon, i, G_0)$ be a groupoid. A pair of nonempty subsets (H, H_0) where $H \subseteq G$ and $H_0 \subseteq G_0$, is called *subgroupoid* of G , if:

- (1) $\alpha(H) = H_0$ and $\beta(H) = H_0$;
 - (2) H is closed under partially multiplication and inversion, that is:
- (i) $\forall x, y \in H$ such that $(x, y) \in G_{(2)} \implies x \cdot y \in H$; (ii) $\forall x \in H \implies x^{-1} \in H$. \square

Example 2.1. (i) A nonempty set G_0 is a groupoid over G_0 , called the *null groupoid*. For this, we take $\alpha = \beta = \varepsilon = i = Id_{G_0}$ and $u \cdot u = u$ for all $u \in G_0$.

(ii) A group G having e as unity is a $\{e\}$ -groupoid with respect to structure functions: $\alpha(x) = \beta(x) := e$; $\varepsilon : \{e\} \rightarrow G$, $\varepsilon(e) := e$, $G_{(2)} = G \times G$, $m(x, y) := xy$ and $i : G \rightarrow G$, $i(x) := x^{-1}$. Conversely, a groupoid with one unit is a group.

(iii) The Cartesian product $G := X \times X$ has a structure of groupoid over X by taking the structure functions as follows: $\bar{\alpha}(x, y) := x$, $\bar{\beta}(x, y) := y$; the elements (x, y) and (y', z) are composable in $G := X \times X$ iff $y' = y$ and we define $(x, y) \cdot (y, z) = (x, z)$, the inclusion map $\bar{\varepsilon} : X \rightarrow X \times X$ is given by $\bar{\varepsilon}(x) := (x, x)$ and the inverse of (x, y) is defined by $(x, y)^{-1} := (y, x)$. This is called the *pair groupoid* associated to set X . Its unit set is $\varepsilon(X) = \{(x, x) \in X \times X | x \in X\}$. The isotropy group $G(x)$ at $x \in X$ is the null group $\{(x, x)\}$.

(iv) For the groupoids $(G, \alpha_G, \beta_G, m_G, \varepsilon_G, i_G, G_0)$ and $(K, \alpha_K, \beta_K, m_K, \varepsilon_K, i_K, K_0)$, one construct the groupoid $(G \times K, G_0 \times K_0)$ with the structure functions given by: $\alpha_{G \times K}(g, k) = (\alpha_G(g), \alpha_K(k))$; $\beta_{G \times K}(g, k) = (\beta_G(g), \beta_K(k))$; $m_{G \times K}((g, k), (g', k')) = (m_G(g, g'), m_K(k, k'))$, $(\forall) (g, g') \in G_{(2)}, (k, k') \in K_{(2)}$; $\varepsilon_{G \times K}(u, v) = (\varepsilon_G(u), \varepsilon_K(v))$, $(\forall) u \in G_0, v \in K_0$ and $i_{G \times K}(g, k) = (i_G(g), i_K(k))$.

This groupoid is called the *direct product* of (G, G_0) and (K, K_0) . \square

Definition 2.3. ([12]) A *morphism of groupoids* or *groupoid morphism* from (G, G_0) into (G', G'_0) is a pair (f, f_0) , where $f : G \rightarrow G'$ and $f_0 : G_0 \rightarrow G'_0$ such that the following conditions hold:

- (1) $\alpha' \circ f = f_0 \circ \alpha$, $\beta' \circ f = f_0 \circ \beta$;
- (2) $f(m(x, y)) = m'(f(x), f(y))$ for all $(x, y) \in G_{(2)}$. \square

A groupoid morphism $(f, Id_{G_0}) : (G, G_0) \rightarrow (G', G_0)$ is called G_0 -*morphism of groupoids*. A groupoid morphism $(f, f_0) : (G, G_0) \rightarrow (G', G'_0)$ such that f and f_0 are bijective maps, is called *isomorphism of groupoids*.

Proposition 2.3. ([11]) If $(f, f_0) : (G, G_0) \rightarrow (G', G'_0)$ is a groupoid morphism, then:

$$f \circ \varepsilon = \varepsilon' \circ f_0 \quad \text{and} \quad f \circ i = i' \circ f. \quad \square$$

For more details on groupoids and their applications, see [1, 12, 15, 9]. Special properties of some classes of Brandt groupoids are presented in [10] and [14].

3 Basic properties in category of group-groupoids

In this section we refer to notion of group-groupoid [4].

A group structure on a nonempty set is regarded as an universal algebra determined by a binary operation, a nullary operation and an unary operation.

Let $(G, \alpha, \beta, m, \varepsilon, i, G_0)$ be a groupoid. We suppose that on G is defined a group structure $\omega : G \times G \rightarrow G$, $(x, y) \rightarrow \omega(x, y) := x \oplus y$. Also, we suppose that on G_0 is defined a group structure $\omega_0 : G_0 \times G_0 \rightarrow G_0$, $(u, v) \rightarrow \omega_0(u, v) := u \oplus v$. The unit element of G (resp., G_0) is e (resp., e_0); that is $\nu : \{\lambda\} \rightarrow G$, $\lambda \rightarrow \nu(\lambda) := e$ (resp., $\nu_0 : \{\lambda\} \rightarrow G_0$, $\lambda \rightarrow \nu_0(\lambda) := e_0$) (here $\{\lambda\}$ is a singleton). The inverse of $x \in G$ (resp., $u \in G_0$) is denoted by \bar{x} (resp., \bar{u}); that is $\sigma : G \rightarrow G$, $x \rightarrow \sigma(x) := \bar{x}$ (resp., $\sigma_0 : G_0 \rightarrow G_0$, $u \rightarrow \sigma_0(u) := \bar{u}$).

Definition 3.1. ([4]) A *group-groupoid* or *\mathcal{G} -groupoid*, is a groupoid (G, G_0) such that the following conditions hold:

- (i) (G, ω, ν, σ) and $(G_0, \omega_0, \nu_0, \sigma_0)$ are groups.
- (ii) The maps $(\omega, \omega_0) : (G \times G, G_0 \times G_0) \rightarrow (G, G_0)$, $(\nu, \nu_0) : (\{\lambda\}, \{\lambda\}) \rightarrow (G, G_0)$ and $(\sigma, \sigma_0) : (G, G_0) \rightarrow (G, G_0)$ are groupoid morphisms. \square

We shall denote a group-groupoid by $(G, \alpha, \beta, m, \varepsilon, i, \oplus, G_0)$.

Proposition 3.1. *If $(G, \alpha, \beta, m, \varepsilon, i, \oplus, G_0)$ is a group-groupoid, then:*

- (i) *the multiplication m and binary operation ω are compatible, that is:*

$$(x \cdot y) \oplus (z \cdot t) = (x \oplus z) \cdot (y \oplus t), \quad (\forall)(x, y), (z, t) \in G_{(2)}; \quad (3.1)$$

- (ii) $\alpha, \beta : (G, \oplus) \rightarrow (G_0, \oplus)$, $i : (G, \oplus) \rightarrow (G, \oplus)$ and $\varepsilon : (G_0, \oplus) \rightarrow (G, \oplus)$ are morphisms of groups; i.e., for all $x, y \in G$ and $u, v \in G_0$, we have:

$$\alpha(x \oplus y) = \alpha(x) \oplus \alpha(y), \quad \beta(x \oplus y) = \beta(x) \oplus \beta(y), \quad i(x \oplus y) = i(x) \oplus i(y); \quad (3.2)$$

$$\varepsilon(u \oplus v) = \varepsilon(u) \oplus \varepsilon(v); \quad (3.3)$$

- (iii) *the multiplication m and the unary operation σ are compatible, that is:*

$$\sigma(x \cdot y) = \sigma(x) \cdot \sigma(y), \quad (\forall)(x, y) \in G_{(2)}. \quad (3.4)$$

Proof. By Definition 2.3, since (ω, ω_0) is a groupoid morphism it follows that:

- (1) $\alpha \circ \omega = \omega_0 \circ (\alpha \times \alpha)$ and $\beta \circ \omega = \omega_0 \circ (\beta \times \beta)$;
- (2) $\omega(m_{G \times G}((x, y), (z, t))) = m_G(\omega(x, z), \omega(y, t))$, $(\forall) (x, y), (z, t) \in G_{(2)}$.

- (i) We have

$$\omega(m_{G \times G}((x, y), (z, t))) = \omega(m_G(x, y), m_G(z, t)) = \omega(x \cdot y, z \cdot t) = (x \cdot y) \oplus (z \cdot t) \quad \text{and} \\ m_G(\omega(x, z), \omega(y, t)) = m_G(x \oplus z, y \oplus t) = (x \oplus z) \cdot (y \oplus t).$$

Using (2) one obtains $(x \cdot y) \oplus (z \cdot t) = (x \oplus z) \cdot (y \oplus t)$, and (3.1) holds.

- (ii) For each $(x, y) \in G \times G$, we have

$$\alpha(\omega(x, y)) = \alpha(x \oplus y) \quad \text{and} \quad \omega_0((\alpha \times \alpha)(x, y)) = \omega_0(\alpha(x), \alpha(y)) = \alpha(x) \oplus \alpha(y).$$

According to the first equality (1), it follows $\alpha(x \oplus y) = \alpha(x) \oplus \alpha(y)$, and the first relation of (3.2) holds. Similarly, we prove that the second relation of (3.2) holds.

Since (ω, ω_0) is a groupoid morphism, by Proposition 2.3, it follows

- (3) $\omega \circ (\varepsilon \times \varepsilon) = \varepsilon \circ \omega_0$ and $i \circ \omega = \omega \circ (i \times i)$.

For each $(x, y) \in G \times G$, we have

$$i(\omega(x, y)) = i(x \oplus y) \quad \text{and} \quad \omega((i \times i)(x, y)) = \omega(i(x), i(y)) = i(x) \oplus i(y).$$

Using now the second equality (3), it follows $i(x \oplus y) = i(x) \oplus i(y)$, and the third relation (3.2) holds. For each $(u, v) \in G_0 \times G_0$, we have

$$\omega((\varepsilon \times \varepsilon)(u, v)) = \omega(\varepsilon(u), \varepsilon(v)) = \varepsilon(u) \oplus \varepsilon(v) \quad \text{and} \quad \varepsilon(\omega_0(u, v)) = \varepsilon(u \oplus v).$$

From the first equality (3), it follows $\varepsilon(u \oplus v) = \varepsilon(u) \oplus \varepsilon(v)$. Hence, (3.3) holds.

(iii) Since (σ, σ_0) is a groupoid morphism, for all $(x, y) \in G_{(2)}$ we have $\sigma(m(x, y)) = m(\sigma(x), \sigma(y))$; i.e., $\sigma(x \cdot y) = \sigma(x) \cdot \sigma(y)$. Hence (3.4) holds. \square

The relation (3.1) (resp., (3.4)) is called the *interchange law* between groupoid multiplication m and group operation ω (resp., σ).

From Proposition 3.1 follows the following corollary.

Corollary 3.1. *Let $(G, \alpha, \beta, m, \varepsilon, i, \oplus, G_0)$ be a \mathcal{G} -groupoid. Then:*

(i) *The source and target $\alpha, \beta : G \rightarrow G_0$ are group epimorphisms, and*

$$\alpha(e) = \beta(e) = e_0, \quad \alpha(\bar{x}) = \overline{\alpha(x)} \quad \text{and} \quad \beta(\bar{x}) = \overline{\beta(x)}, \quad (\forall) x \in G; \quad (3.5)$$

(ii) *The inclusion map $\varepsilon : G_0 \rightarrow G$ is a group monomorphism, and*

$$\varepsilon(e_0) = e, \quad \varepsilon(\bar{u}) = \overline{\varepsilon(u)}, \quad (\forall) u \in G_0; \quad (3.6)$$

(iii) *The inversion $i : G \rightarrow G$ is a group automorphism, and*

$$i(e) = e, \quad i(\bar{x}) = \overline{i(x)}, \quad (\forall) x \in G; \quad (3.7)$$

(iv) *For all $x, y \in G$, we have*

$$\sigma(x \oplus y) = \sigma(y) \oplus \sigma(x) \quad \text{and} \quad \sigma(\sigma(x)) = x. \quad \square \quad (3.8)$$

We say that the group-groupoid $(G, \alpha, \beta, m, \varepsilon, i, \oplus, G_0)$ is a *commutative group-groupoid*, if the groups G and G_0 are commutative.

Corollary 3.2. *Let $(G, \alpha, \beta, m, \varepsilon, i, \oplus, G_0)$ be a commutative \mathcal{G} -groupoid. Then:*

$$\overline{x \oplus y} = \bar{x} \oplus \bar{y}, \quad (\forall) x, y \in G.$$

Proof. It is an immediate consequence of the relation (3.8). \square

Proposition 3.2. *If $(G, \alpha, \beta, m, \varepsilon, i, \oplus, G_0)$ is a \mathcal{G} -groupoid, then:*

$$e \cdot y = y, \quad (\forall) y \in \alpha^{-1}(e_0) \quad \text{and} \quad x \cdot e = x, \quad (\forall) x \in \beta^{-1}(e_0); \quad (3.9)$$

$$x \cdot (y \oplus t) = x \cdot y \oplus t, \quad (\forall) (x, y) \in G_{(2)} \quad \text{and} \quad t \in \alpha^{-1}(e_0); \quad (3.10)$$

$$(x \oplus z) \cdot y = x \cdot y \oplus z, \quad (\forall) (x, y) \in G_{(2)} \quad \text{and} \quad z \in \beta^{-1}(e_0). \quad (3.11)$$

Proof. If $y \in \alpha^{-1}(e_0)$, then $\alpha(y) = e_0$. We have $\beta(\varepsilon(e_0)) = e_0$, since $\beta \circ \varepsilon = Id_{G_0}$. So $(\varepsilon(e_0), y) \in G_{(2)}$. Using (3.6) and the condition (G2) from Definition 2.1, one obtains $e \cdot y = \varepsilon(e_0) \cdot y = \varepsilon(\alpha(y)) \cdot y = y$. Hence the first relation of (3.9) holds. Similarly, we verify the second equality of (3.9).

For to prove the relation (3.10) we apply the interchange law (3.1) and (3.9). Indeed, if in (3.1) we replace z with e , one obtains $(x \cdot y) \oplus (e \cdot t) = (x \oplus e) \cdot (y \oplus t)$, for all $(x, y), (e, t) \in G_{(2)}$. It follows $(x \cdot y) \oplus t = x \cdot (y \oplus t)$, since $x \oplus e = x, \beta(e) = e_0$ and $t \in \alpha^{-1}(e_0)$. Hence, the relation (3.10) holds.

Similarly, if in (3.1) we replace t with e , one obtains $(x \cdot y) \oplus (z \cdot e) = (x \oplus y) \cdot (y \oplus e)$, for all $(x, y), (z, e) \in G_{(2)}$. It follows $(x \cdot y) \oplus z = (x \oplus z) \cdot y$, since $y \oplus e = y, \alpha(e) = e_0$ and $z \in \beta^{-1}(e_0)$. Hence, the relation (3.11) holds. \square

Proposition 3.3. ([3]) *If $(G, \alpha, \beta, m, \varepsilon, i, \oplus, G_0)$ is a \mathcal{G} -groupoid, then:*

$$x \cdot y = x \oplus \overline{\varepsilon(\beta(x))} \oplus y, \quad (\forall)(x, y) \in G_{(2)}; \quad (3.12)$$

$$x^{-1} = \varepsilon(\alpha(x)) \oplus \bar{x} \oplus \varepsilon(\beta(x)), \quad (\forall)x \in G. \quad (3.13)$$

Proof. Fix $x, y \in G$ and introduce the notations $u := \alpha(x)$, $v := \beta(x)$ and $w := \beta(y)$. Consider $(x, y) \in G_{(2)}$. Then $\beta(x) = \alpha(y) = v$. Since \oplus is associative, we have

$$(1) \quad x \cdot y = ((x \oplus \overline{\varepsilon(v)}) \oplus \varepsilon(v)) \cdot (e \oplus y).$$

Using the fact that β is a group morphism and (3.5) we have

$\beta(x \oplus \overline{\varepsilon(v)}) = \beta(x) \oplus \beta(\overline{\varepsilon(v)}) = v \oplus \overline{\beta(\varepsilon(v))} = v \oplus \bar{v} = e_0 = \alpha(e)$. Then the product $(x \oplus \overline{\varepsilon(v)}) \cdot e$ is defined and $x \oplus \overline{\varepsilon(v)} \in \beta^{-1}(e_0)$. Applying (3.9) one obtains $(x \oplus \overline{\varepsilon(v)}) \cdot e = x \oplus \varepsilon(v)$. On the other hand, we have $\varepsilon(v) \cdot y = \varepsilon(\alpha(y)) \cdot y = y$. Applying now the interchange law (3.1), from (1) implies that

$$x \cdot y = ((x \oplus \overline{\varepsilon(v)}) \cdot e) \oplus (\varepsilon(v) \cdot y) \Rightarrow x \cdot y = x \oplus \overline{\varepsilon(v)} \oplus y \Rightarrow x \cdot y = x \oplus \overline{\varepsilon(\beta(x))}.$$

For the prove (3.13) denote $a := \varepsilon(\alpha(x)) \oplus \bar{x} \oplus \varepsilon(\beta(x))$. Then $a := \varepsilon(u) \oplus \bar{x} \oplus \varepsilon(v)$. Applying the fact that α is a group morphism and (3.5), we have

$\alpha(a) = u \oplus \overline{\alpha(x)} \oplus v = u \oplus \bar{u} \oplus v = v$. Then the product $x \cdot a$ is defined. We have

$$(2) \quad x \cdot a = (e \oplus x) \cdot ((\varepsilon(u) \oplus \bar{x}) \oplus \varepsilon(v)).$$

Applying the interchange law (3.1) and (3.9), from (2) we have

$$x \cdot a = (e \cdot (\varepsilon(u) \oplus \bar{x})) \oplus (x \cdot \varepsilon(v)) \Rightarrow x \cdot a = \varepsilon(u) \oplus \bar{x} \oplus x \Rightarrow x \cdot a = \varepsilon(u).$$

Hence, $x \cdot a = \varepsilon(\alpha(x))$. Similarly, we verify that $a \cdot x = \varepsilon(\beta(x))$. Then $a = x^{-1}$ and the relation (3.13) holds. \square

Corollary 3.3. *If $(G, \alpha, \beta, m, \varepsilon, i, \oplus, G_0)$ is a \mathcal{G} -groupoid, then:*

$$x \cdot y = x \oplus y \quad \text{and} \quad x^{-1} = \bar{x}, \quad (\forall)x, y \in G(e_0). \quad (3.14)$$

Proof. Let $x, y \in G(e_0)$. We have $(x, y) \in G_{(2)}$, since $\beta(x) = \alpha(y) = e_0$. From (3.12), we have $x \cdot y = x \oplus y$, since $\varepsilon(\beta(x)) = \varepsilon(e_0) = e$ and $\bar{e} = e$. Hence, the first equality from (3.14) holds. Replacing in (3.13), $\varepsilon(\alpha(x)) = \varepsilon(\beta(x)) = e$ one obtains $x^{-1} = \bar{x}$. Therefore, the second equality from (3.14) holds. \square

Theorem 3.1. *Let $(G, \alpha, \beta, m, \varepsilon, i, G_0)$ be a groupoid. If the following conditions are satisfied:*

- (i) (G, \oplus) and (G_0, \oplus) are groups;
 - (ii) $\alpha, \beta : (G, \oplus) \rightarrow (G_0, \oplus)$, $\varepsilon : (G_0, \oplus) \rightarrow (G, \oplus)$ and $i : (G, \oplus) \rightarrow (G, \oplus)$ are morphisms of groups;
 - (iii) the interchange law (3.1) between the operations m and ω holds,
- then $(G, \alpha, \beta, m, \varepsilon, i, \oplus, G_0)$ is a group-groupoid.

Proof. By hypothesis, the condition (i) from Definition 3.1 is verified. It remains to prove that the condition (ii) holds.

(a) We prove that $(\omega, \omega_0) : (G \times G, G_0 \times G_0) \rightarrow (G, G_0)$ is a morphism of groupoids. Since α is a morphism of groups, it follows $\alpha(x \oplus y) = \alpha(x) \oplus \alpha(y)$, for all $x, y \in G$. Then $\alpha(\omega(x, y)) = \omega_0(\alpha(x), \alpha(y))$, and it follows $\alpha(\omega(x, y)) = \omega_0((\alpha \times \alpha)(x, y))$; i.e., $\alpha \circ \omega = \omega_0 \circ (\alpha \times \alpha)$. Similarly, we prove that $\beta \circ \omega = \omega_0 \circ (\beta \times \beta)$. Hence the condition (i) from Definition 2.3 is satisfied.

We suppose that the interchange law (3.1) holds. Then, for all (x, y) and (z, t) in $G_{(2)}$ we have $(x \cdot y) \oplus (z \cdot t) = (x \oplus z) \cdot (y \oplus t)$. From the last equality it follows $m(x, y) \oplus m(z, t) = \omega(x, z) \cdot \omega(y, t) \Rightarrow \omega(m(x, y), m(z, t)) = m(\omega(x, z), \omega(y, t))$.

Then $\omega(m_{G \times G}((x, y), (z, t))) = m(\omega(x, z), \omega(y, t))$, and the condition (ii) from Definition 2.3 holds. Hence, (ω, ω_0) is a groupoid morphism.

(b) We prove that (ν, ν_0) is a morphism of groupoids (here $\{\lambda\}$ is regarded as null groupoid with the structure functions $\alpha'_0, \beta'_0, \varepsilon'_0, i'_0$ and multiplication m'_0). Since α and ε are group morphisms, we have $\alpha(e) = e_0$ and $\varepsilon(e_0) = e$. From $\alpha(\nu(\lambda)) = \alpha(e) = e_0$ and $\nu_0(\lambda) = e_0$, it follows $\alpha \circ \nu = \nu_0 \circ Id$. Similarly, we have $\beta \circ \nu = \nu_0 \circ Id$. Also, we have $\nu(m'_0(\lambda, \lambda)) = \nu(\lambda \cdot \lambda) = \nu(\lambda) = e$ and

$m(\nu(\lambda), \nu(\lambda)) = e \cdot e = \varepsilon(e_0) \cdot e = \varepsilon(\alpha(e)) \cdot e = e$. Then, $\nu(m'_0(\lambda, \lambda)) = m(\nu(\lambda), \nu(\lambda))$. Hence, the pair (ν, ν_0) is a groupoid morphism.

(c) We prove that (σ, σ_0) is a groupoid morphism. Applying (3.6) we have $\alpha(\sigma(x)) = \alpha(\bar{x}) = \overline{\alpha(x)}$ and $\sigma_0(\alpha(x)) = \overline{\alpha(x)}$. Then $\alpha \circ \sigma = \sigma_0 \circ \alpha$. Similarly, we have $\beta \circ \sigma = \sigma_0 \circ \beta$. We shall prove that:

(1) $\overline{x \cdot y} = \bar{x} \cdot \bar{y}$, $(\forall) (x, y) \in G_{(2)}$.

From $(x, y) \in G_{(2)}$ we have $\beta(x) = \alpha(y)$. Then $\overline{\beta(x)} = \overline{\alpha(y)}$, and it follows $\beta(\bar{x}) = \alpha(\bar{y})$. Therefore $(\bar{x}, \bar{y}) \in G_{(2)}$. Using now (3.1) one obtains

(2) $(x \cdot y) \oplus (\bar{x} \cdot \bar{y}) = (x \oplus \bar{x}) \cdot (y \oplus \bar{y})$ and $(\bar{x} \cdot \bar{y}) \oplus (x \cdot y) = (\bar{x} \oplus x) \cdot (\bar{y} \oplus y)$.

Using the relations $a \oplus \bar{a} = \bar{a} \oplus a = e$, and $e \cdot e = e$, from (2), we have

(3) $(x \cdot y) \oplus (\bar{x} \cdot \bar{y}) = e$ and $(\bar{x} \cdot \bar{y}) \oplus (x \cdot y) = e$.

From (3) one obtains that the equality (1) holds.

The relation (1) is equivalently with

$\sigma(x \cdot y) = \sigma(x) \cdot \sigma(y) \Leftrightarrow \sigma(m_G(x, y)) = m_G(\sigma(x), \sigma(y))$.

Hence, (σ, σ_0) is a groupoid morphism. \square

According to Proposition 3.1 and Theorem 3.1 we give a new definition (Definition 3.2) for the notion of group-groupoid (this is equivalent with Definition 3.1).

Definition 3.2. A *group-groupoid* is a groupoid $(G, \alpha, \beta, m, \varepsilon, i, G_0)$ such that the following conditions are satisfied:

(i) (G, \oplus) and (G_0, \oplus) are groups;

(ii) $\alpha, \beta : (G, \oplus) \rightarrow (G_0, \oplus)$, $\varepsilon : (G_0, \oplus) \rightarrow (G, \oplus)$ and $i : (G, \oplus) \rightarrow (G, \oplus)$ are morphisms of groups;

(iii) the interchange law (3.1) between the operations m and \oplus holds. \square

If in Definition 3.2, we consider $G_0 \subseteq G$ and $\varepsilon : G_0 \rightarrow G$ is the inclusion map, then $(G, \alpha, \beta, m, i, \oplus, G_0)$ is a group-groupoid and we say that it is a *group- G_0 -groupoid*.

Example 3.1. (i) Let G_0 be a group. Then G_0 has a structure of null groupoid over G_0 (see Example 2.1(i)). We have that $G = G_0$ and $\alpha, \beta, \varepsilon, i$ are morphisms of groups. It is easy to prove that the interchange law (3.1) is verified. Then G_0 is a group-groupoid, called the *null group-groupoid* associated to group G_0 .

(ii) A commutative group (G, \oplus) having $\{e\}$ as unity may be considered to be a $\{e\}$ -groupoid (see Example 2.1(ii)). In this case, $m = \oplus$. We have that (G, \oplus) and $G_0 = \{e\}$ are groups. It is easy to see that $\alpha, \beta, \varepsilon$ and i are morphisms of groups. It remains to verify that (3.1) holds. Indeed, for $x, y, z, t \in G$ we have $(x \oplus y) \oplus (z \oplus t) = (x \oplus z) \oplus (y \oplus t)$, since the operation \oplus is associative and commutative. Hence $(G, \alpha, \beta, m, \varepsilon, i, \oplus, \{e\})$ is a group-groupoid called *group-groupoid with a single unit* associated to commutative group (G, \oplus) . Therefore, *each commutative group G can be regarded as a commutative group $-\{e\}$ -groupoid*. \square

Example 3.2. Let (G, \oplus) be a group and $(G \times G, \overline{\alpha}, \overline{\beta}, \overline{m}, \overline{\varepsilon}, \overline{i}, G)$ the pair groupoid associated to G (see Example 2.1(iii)). We have that $G \times G$ is a group endowed with operation $(x_1, x_2) \oplus (y_1, y_2) := (x_1 \oplus y_1, x_2 \oplus y_2)$, for all $x_1, x_2, y_1, y_2 \in G$. It is easy to check that $\overline{\alpha}, \overline{\beta}, \overline{\varepsilon}$ and \overline{i} are group morphisms. Therefore, the conditions (i) and (ii) from Definition 3.2 are satisfied. For to prove that the condition (iii) is verified, we consider $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2), t = (t_1, t_2)$ from $G \times G$ such that $\overline{\beta}(x) = \overline{\alpha}(y)$ and $\overline{\beta}(z) = \overline{\alpha}(t)$. Then $x_2 = y_1$ and $z_2 = t_1$. It follows $y = (x_2, y_2), t = (z_2, t_2), x \cdot y = (x_1, y_2)$ and $z \cdot t = (z_1, t_2)$. We have $(x \cdot y) \oplus (z \cdot t) = (x_1, y_2) \oplus (z_1, t_2) = (x_1 \oplus z_1, y_2 \oplus t_2)$ and $(x \oplus z) \cdot (y \oplus t) = (x_1 \oplus z_1, x_2 \oplus z_2) \cdot (x_2 \oplus z_2, y_2 \oplus t_2) = (x_1 \oplus z_1, y_2 \oplus t_2)$. Then, $(x \cdot y) \oplus (z \cdot t) = (x \oplus z) \cdot (y \oplus t)$ and so (3.1) holds. Hence $G \times G$ is a group-groupoid called the *group-pair groupoid* associated to group G . \square

Example 3.3. Consider the groups $(\mathbf{R}^2, +)$ and $(\mathbf{R}, +)$. For (G, G_0) where $G := \mathbf{R}^2$ and $G_0 := \mathbf{R}$, we define the structure functions $\alpha, \beta : \mathbf{R}^2 \rightarrow \mathbf{R}, \varepsilon : \mathbf{R} \rightarrow \mathbf{R}^2$ and $i : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ as follows: $\alpha(x_1, x_2) := x_1 + 2x_2, \beta(x_1, x_2) := x_1 + x_2, \varepsilon(x_1) := (x_1, 0)$ and $i(x_1, x_2) := (x_1 + 3x_2, -x_2)$, for all $x_1, x_2 \in \mathbf{R}$.

Let $G_{(2)} := \{(x_1, x_2), (y_1, y_2) \in \mathbf{R}^2 \times \mathbf{R}^2 \mid x_2 = -x_1 + y_1 + 2y_2\}$ be the set of composable pairs. The multiplication $m : G_{(2)} \rightarrow G$ is given by:

$$(x_1, x_2) \cdot (y_1, y_2) := (x_1 - 2y_2, x_2 + y_2), \quad \text{if } x_2 = -x_1 + y_1 + 2y_2.$$

It is easy to check that the above structure functions determine on G a structure of a groupoid over G_0 . Also, the maps $\alpha, \beta, \varepsilon$ and i are group morphisms. Therefore, the conditions (i) and (ii) from the Definition 3.2 hold.

We consider $x, y, z, t \in \mathbf{R}^2$ such that the products $x \cdot y$ and $z \cdot t$ are defined. Then $x = (x_1, x_2), y = (y_1, y_2), z = (z_1, z_2), t = (t_1, t_2)$ such that $x_2 = -x_1 + y_1 + 2y_2$ and $z_2 = -z_1 + t_1 + 2t_2$. We have $x \cdot y = (x_1 - 2y_2, x_2 + y_2), z \cdot t = (z_1 - 2t_2, z_2 + t_2)$. Then $(x \cdot y) + (z \cdot t) = (x_1 - 2y_2 + z_1 - 2t_2, x_2 + y_2 + z_2 + t_2)$ and $(x + z) \cdot (y + t) = (x_1 + z_1 - 2(y_2 + t_2), x_2 + z_2 + y_2 + t_2)$.

Hence, $(x \cdot y) + (z \cdot t) = (x + z) \cdot (y + t)$ and the interchange law (3.1) holds. Therefore, $(\mathbf{R}^2, \alpha, \beta, m, \varepsilon, i, \mathbf{R})$ is a commutative group-groupoid.

Let us we consider the Euclidean plane \mathbf{R}^2 with the Cartesian coordinate system Ox_1x_2 . The α -fibres $\alpha^{-1}(u)$ for $u \in \mathbf{R}$ are represented by parallel straight lines of

equation $x_1 + 2x_2 - u = 0$. Also, the β -fibres $\beta^{-1}(v)$ for $v \in \mathbf{R}$ are represented by parallel straight lines of equation $x_1 + x_2 - v = 0$.

Let be the points A_1, A_2, A_3, A_4 associated to elements $\varepsilon(\beta(x)), x, x \cdot y, y \in G$, for $\beta(x) = \alpha(y)$. Then $A_1(b + 2c, 0), A_2(a, -a + b + 2c), A_3(a - 2c, -a + b + 3c)$ and $A_4(b, c)$. We have that: *the simple quadrilateral $A_1A_2A_3A_4$ is a parallelogram.*

Indeed, the slope of line through A_1 and A_4 is $m_{A_1A_4} = -\frac{1}{2}$ and the distance from A_1 and A_4 is $d(A_1, A_4) = |c|\sqrt{5}$. Also, $m_{A_2A_3} = -\frac{1}{2}$ and $d(A_2, A_3) = |c|\sqrt{5}$.

Let be the points B_1, B_2, B_3, B_4 associated to $\varepsilon(\alpha(x)), x, \varepsilon(\beta(x)), x^{-1} \in G$. We have $B_1(x_1 + 2x_2, 0), B_2(x_1, x_2), B_3(x_1 + x_2, 0)$ and $B_4(x_1 + 3x_2, -x_2)$. Then: *the simple quadrilateral $B_1B_2B_3B_4$ is a parallelogram.*

Indeed, we have $m_{B_1B_2} = m_{B_3B_4} = -\frac{1}{2}$ and $d(B_1, B_2) = d(B_3, B_4) = |x_2|\sqrt{5}$. \square

Definition 3.3. Let $(G, \alpha, \beta, m, \varepsilon, i, \oplus, G_0)$ be a group-groupoid. A subgroupoid (H, H_0) of the groupoid (G, G_0) is called a *group-subgroupoid* of (G, G_0) , if H and H_0 are subgroups in G and G_0 , respectively.

If $H_0 = G_0$ we say that H is a *group $-G_0$ -subgroupoid* of (G, G_0) . \square

According to Definition 3.3, if (H, H_0) is a group-subgroupoid of (G, G_0) , then the pair (H, H_0) endowed with the restrictions of functions α, β, i and \oplus to H , ε to H_0 and m to $H_{(2)}$, is a group-groupoid, denoted by (H, H_0) . We denote by the same letters the structure functions of H as well as those of G .

Proposition 3.4. Let $(G, \alpha, \beta, m, \varepsilon, i, \oplus, G_0)$ be a group-groupoid. Then:

- (i) The fibres $\alpha^{-1}(e_0)$ and $\beta^{-1}(e_0)$ are subgroups in G .
- (ii) The isotropy group $G(e_0)$ is a group- $\{e_0\}$ -subgroupoid of G .
- (iii) $\varepsilon(G_0)$ and G are group $-G_0$ -subgroupoids of G .
- (iv) $Is(G) := \{x \in G \mid \alpha(x) = \beta(x)\}$ is a group $-G_0$ -subgroupoid of G .

Proof. (i) For all $x, y \in \alpha^{-1}(e_0)$ we have $x \oplus y \in \alpha^{-1}(e_0)$ and $\bar{x} \in \alpha^{-1}(e_0)$. Indeed, applying (3.2) we have $\alpha(x \oplus y) = \alpha(x) \oplus \alpha(y) = e_0 \oplus e_0 = e_0$. Also, using (3.5) it follows $\alpha(\bar{x}) = \alpha(x) = \bar{e_0} = e_0$. Then $\alpha^{-1}(e_0)$ is a subgroup in (G, \oplus) . Similarly, we prove that $\beta^{-1}(e_0)$ is a subgroup in (G, \oplus) .

(ii) It is easy to verify that $G(e_0)$ is a $\{e_0\}$ -subgroupoid. Also, according to (i) we have that $G(e_0)$ is a subgroup of (G, \oplus) , since $G(e_0) = \alpha^{-1}(e_0) \cap \beta^{-1}(e_0)$. Hence, the conditions from Definition 3.3 are satisfied and $G(e_0)$ is a group- $\{e_0\}$ -subgroupoid.

(iii) It is easy to verify that $\varepsilon(G_0)$ and G are group $-G_0$ -subgroupoids.

(iv) Clearly, $\alpha(Is(G)) = \beta(Is(G)) = G_0$. Let $x, y \in Is(G)$ with $(x, y) \in G_{(2)}$. Then $\alpha(x) = \beta(x) = \alpha(y) = \beta(y)$. We have $\alpha(xy) = \alpha(x) = \beta(y) = \beta(xy)$ and $\alpha(x^{-1}) = \beta(x) = \alpha(x) = \beta(x^{-1})$. Hence, $xy, x^{-1} \in Is(G)$ and $Is(G)$ is a subgroupoid. Since α and β are group morphisms, implies that $\alpha(x \oplus y) = \beta(x \oplus y)$ and $\alpha(\bar{x}) = \beta(\bar{x})$. Then, for all $x, y \in Is(G)$ we have $x \oplus y, \bar{x} \in Is(G)$. Therefore, $Is(G)$ is a subgroup in (G, \oplus) . Hence, $Is(G)$ is a group $-G_0$ -subgroupoid. \square

The group-subgroupoid $Is(G)$ is the union of all isotropy groups of G and it is called the *isotropy bundle* of the group-groupoid (G, G_0) .

Definition 3.4. Let $(G_j, \alpha_j, \beta_j, m_j, \varepsilon_j, i_j, \oplus_j, G_{j,0})$, $j = 1, 2$ be two group-groupoids. A groupoid morphism $(f, f_0) : (G_1, G_{1,0}) \rightarrow (G_2, G_{2,0})$ such that f and f_0 are group morphisms, is called *group-groupoid morphism* or *morphism of group-groupoids*.

A group-groupoid morphism of the form $(f, Id_{G_{1,0}}) : (G_1, G_{1,0}) \rightarrow (G_2, G_{1,0})$ is called $G_{1,0}$ -*morphism of group-groupoids*. It is denoted by $f : G_1 \rightarrow G_2$. \square

The category of group-groupoids, denoted by $\mathcal{G}Gpd$, has its objects all group-groupoids (G, G_0) and as morphisms from (G, G_0) to (G', G'_0) the set of all morphisms of group-groupoids.

Example 3.4. Direct product of two group-groupoids. Let given the group-groupoids (G, G_0) and (K, K_0) . We consider the direct product $(G \times K, G_0 \times K_0)$ of the groupoids (G, G_0) and (K, K_0) (see Example 2.1 (iv)). On $G \times K$ and $G_0 \times K_0$ we introduce the usual group operations. These operations are defined by $(g_1, k_1) \oplus_{G \times K} (g_2, k_2) := (g_1 \oplus_G g_2, k_1 \oplus_K k_2)$, $(\forall) g_1, g_2 \in G, k_1, k_2 \in K$ and $(u_1, v_1) \oplus_{G_0 \times K_0} (u_2, v_2) := (u_1 \oplus_{G_0} u_2, v_1 \oplus_{K_0} v_2)$, $(\forall) u_1, u_2 \in G_0, v_1, v_2 \in K_0$.

By a direct computation we prove that the conditions from Definition 3.2 are satisfied. Then $(G \times K, G_0 \times K_0)$ is a group-groupoid, called the *direct product of group-groupoids* (G, G_0) and (K, K_0) . The canonical projections $pr_G : G \times K \rightarrow G$ and $pr_K : G \times K \rightarrow K$ are morphisms of group-groupoids. \square

Proposition 3.5. Let $(G, \alpha, \beta, m, \varepsilon, i, \oplus, G_0)$ be a group-groupoid. The anchor map $(\alpha, \beta) : G \rightarrow G_0 \times G_0$ is a G_0 -morphism of group-groupoids from the group-groupoid (G, G_0) into the group-pair groupoid $(G_0 \times G_0, \bar{\alpha}, \bar{\beta}, \bar{m}, \bar{\varepsilon}, \bar{i}, \oplus, G_0)$.

Proof. We denote $(\alpha, \beta) := f$. Then $f(x) = (\alpha(x), \beta(x))$, for all $x \in G$. We prove that $\bar{\alpha} \circ f = \alpha$. Indeed, for all $x \in G$ we have $(\bar{\alpha} \circ f)(x) = \bar{\alpha}(\alpha(x), \beta(x)) = \alpha(x)$. Therefore, $\bar{\alpha} \circ f = \alpha$. Similarly, we verify that $\bar{\beta} \circ f = \beta$.

For $(x, y) \in G_{(2)}$ we have $f(x \cdot y) = (\alpha(x \cdot y), \beta(x \cdot y)) = (\alpha(x), \beta(y))$ and $\bar{m}(f(x), f(y)) = \bar{m}((\alpha(x), \beta(x)), (\alpha(y), \beta(y))) = (\alpha(x), \beta(y))$, since $\beta(x) = \alpha(y)$. Therefore, $f(x \cdot y) = \bar{m}(f(x), f(y))$. Hence, f is a G_0 -morphism of groupoids.

Let $x, y \in G$. Since α, β are group morphisms, we have $f(x \oplus y) = (\alpha(x) \oplus \alpha(y), \beta(x) \oplus \beta(y)) = f(x) \oplus f(y)$, i.e. f is a morphism of groups. Hence f is a G_0 -morphism of group-groupoids. \square

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